Ergodic Theory and Measured Group Theory
Lecture 6

Classical pointwise ergodic theorem (Birkhoff 1931). Let T be a pmp transformation on a st. prob. sp. (X,μ). If T is ergodic then for every $f \in L^1(X,\mu)$,

$$\lim_{n\to\infty} A_n^T f(x) = \int f \, d\mu, \quad \text{a.e. } x \in X,$$

where $A_n^T f(x) = \frac{1}{n+1}\left(f(x) + f(Tx) + \dots + f(T^n x)\right)$ = average of f over $I_n^T x$.

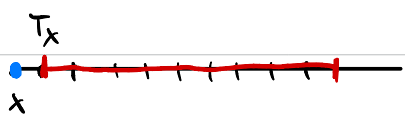
Proof of the ptwise ergodic theorem (Keane-Petersen, Ts.).

(1) Bridge lemma. $\int_X f \, d\mu = \int_X A_n^T f \, d\mu \quad \forall n \in \mathbb{N}$.

To prove that the limit exists and is $= \int f \, d\mu$, we'll show that $\limsup \leq \int f \, d\mu$ and $\liminf \geq \int f \, d\mu$. So it makes sense to look at limsup and liminf.

(2) $\limsup_n A_n f$, $\liminf_n A_n f$ are T-invariant.

Proof. We need to show that limsup



$$A_n f(x) = \frac{1}{n+1} \cdot f(x) + \frac{n}{n+1} \cdot A_{n-1} f(Tx).$$

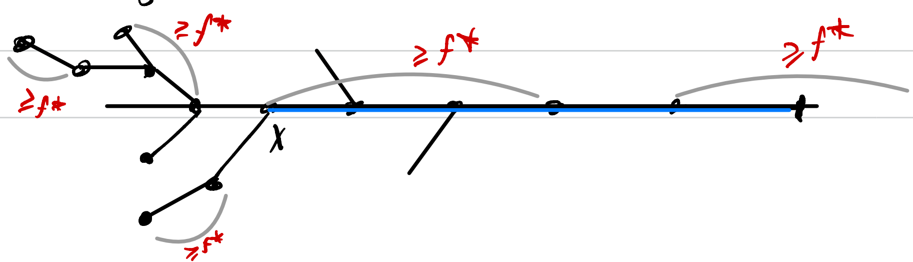
is constant on every orbit, for which enough to fix x and show that limsup at x = limsup at Tx. (Because if y & z are in the same orbit then $T^k y = T^m z$.)

$A_n f(x) = \frac{1}{n+1} \cdot f(x) + \frac{n}{n+1} \cdot A_{n-1} f(Tx)$, so as $n \to \infty$ $\limsup A_n f(x) = \limsup A_n f(Tx)$. $\square$

Remark. This happened because intervals have a small boundary relative to their size. This property is call Følnerness.

- WLOG, assume $\int f \, d\mu = 0$ (by considering $f - \int f \, d\mu$ instead).
- By ergodicity and (2), $\limsup_n A_n f$ and $\liminf_n A_n f$ are constant a.e.
- Suppose towards a contradiction that $\limsup_n A_n f > 0$, so $f^* := \min \left\{ \frac{1}{2} \limsup_n A_n f, 1 \right\} > 0$.

The vague hope is to cover each orbit by intervals s.t. the average of f on those intervals is $\geq f^*$.

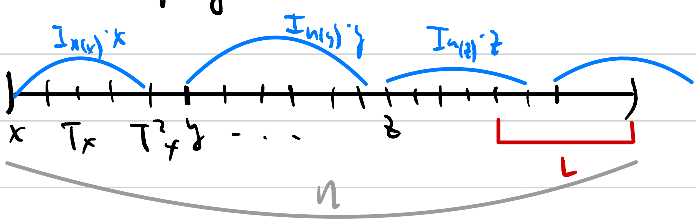


A more reasonable goal is to show that $A_n f > f^* - \varepsilon \ \forall x \in X$ for a fixed n.

- Define $X \to \mathbb{N}$ $x \mapsto$ the least $n = n(x)$ s.t. $A_{n(x)} f(x) \geq f^*$ (such an n exists by the def. of limsup).

- Assume $x \mapsto n(x)$ is bounded.

(3') <u>Bounded tiling lemma.</u> If $x \mapsto n(x)$ is a bounded function $X \to \mathbb{N}$, then $\forall \varepsilon > 0 \ \forall^\infty n \ \forall x \in X$ $(1-\varepsilon)$ fraction of $I_n \cdot x$ is tiled by intervals of the form $I_{n(y)} \cdot y$.



Proof. Let $L \in \mathbb{N}$ be a bound for $x \mapsto n(x)$.

Tiling the interval $I_n \cdot x$ left-to-right, we would possibly leave untiled the points in $\{T^{n-L} x, T^{n-L+1} x, \dots, T^n x\}$.

Thus, only $\leq \frac{L}{n}$ fraction remains untiled so take n large enough to make $L/n < \varepsilon$. $\square$

- Assume f is bounded, i.e. $f \in L^\infty(X,\mu)$. In particular, $f \geq -M$, $M>0$.

- By the bounded tiling lemma, $\forall \varepsilon$ (will be chosen later), $\forall^\infty_n \forall_x$ the interval $I_{n,x}$ is tiled, up to ε fraction, by intervals over which the average of f is $\geq f^*$. Hence:
$$A_n f(x) \geq (1-\varepsilon)\cdot f^* + \varepsilon \cdot (-M) \geq \frac{f^*}{3} \text{ for a small enough } \varepsilon.$$

- By the bridge lemma,
$$0 = \int f \, d\mu = \int A_n f \, d\mu \geq \frac{f^*}{3} \cdot 1 > 0, \text{ a contradiction.}$$

Thus, $\limsup_n A_n f \leq 0$. Similarly, $\liminf_n A_n f \geq 0$, so $\lim_n A_n f = 0$. $\boxtimes$

- We don't assume $x \to n(x)$ is bounded.

<u>Notation.</u> $\forall^\infty_n$:= for all but finitely-many n := $\exists N \, \forall n \geq N$.

$\exists^\infty_n$:= for infinitely-many n := $\forall N \, \exists n \geq N$.

$\forall^\varepsilon_x$:= for all x except for x in a set of measure $\leq \varepsilon$.

:= $\forall x \in B$, where $X \setminus B$ has measure $\leq \varepsilon$.

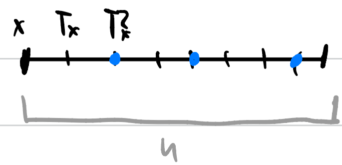
$\exists^\varepsilon_x$:= $\exists$ $\geq\varepsilon$-measured set of x := $\exists B$ of measure $\geq \varepsilon$ s.t. $\forall x \in B \ldots$

(3) <u>Tiling lemma.</u> For any measurable $x \mapsto n(x) : X \to \mathbb{N}$, $\forall \varepsilon > 0$ $\forall^\infty n$ $\forall^\varepsilon x$ the set $I_n \cdot x$ is tiled, up to an $\le \varepsilon$-fraction, by intervals of the form $I_{n(y)} \cdot y$.

Proof. $\forall x\ \exists L\ n(x) \le L$, hence $\exists L\ \forall^\delta x\ n(x) \le L$. (This is because $X = \bigcup_L \{x : n(x) \le L\}$.) Thus, there is $L \in \mathbb{N}$ s.t. the set

$$B := \{x \in X : n(x) > L\}$$

has measure $\le \varepsilon^2/2$. For any $x \in X$,



<u>Claim.</u> $\forall n\ \forall^\varepsilon x \in X\quad A_n \mathbb{1}_B(x) < \frac{\varepsilon}{2}$.

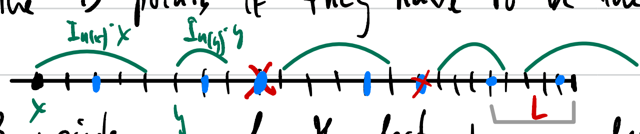
Proof. By the bridge lemma,

$$\frac{\varepsilon^2}{2} \ge \mu(B) = \int \mathbb{1}_B \, d\mu = \int A_n \mathbb{1}_B \, d\mu \ge \int_{A_n \mathbb{1}_B \ge \frac{\varepsilon}{2}} A_n \mathbb{1}_B \, d\mu \ge$$

$$\ge \frac{\varepsilon}{2} \cdot \mu\left(A_n \mathbb{1} \ge \varepsilon/2\right).$$

Hence, $\mu(A_n \mathbb{1}_B \ge \varepsilon/2) \le \varepsilon$. $\square$

Fix an x s.t. $A_n \mathbb{1}_B(x) < \frac{\varepsilon}{2}$. Tile it left-to-right but skipping the B points if they have to be the base of a tile.



We tile all except B points and the last L, hence left out $\le \frac{\varepsilon}{2} + \frac{L}{n}$ fraction. Choose n so that $\frac{L}{n} < \frac{\varepsilon}{2}$. $\square$